

HOMOTOPY, SIMPLE HOMOTOPY AND COMPACTA

STEVE FERRY†

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IN THIS paper, we will answer two questions concerning the geometric homotopy theory of compact spaces. Recall that a map $d: X \rightarrow Y$ is said to be a *homotopy domination* if there is a map $u: Y \rightarrow X$ such that $d \circ u \simeq id_Y$.

QUESTION 1 [6]. *If a compactum is homotopy dominated by a finite CW complex, must it be homotopy equivalent to a finite CW complex?*

Question 1 is related to a series of questions which have been studied over the past 30 yr. Thus, in 1950 J.H.C. Whitehead proved that such a space is homotopy equivalent to an infinite dimensional CW complex. He also asked ([21, p. 108]) whether such a space necessarily had the homotopy type of a finite dimensional CW complex. In 1957, Milnor ([15], p. 273) asked whether an arbitrary space which is homotopy dominated by a finite CW complex must be homotopy equivalent to some finite complex. In 1965 Whitehead's question was answered affirmatively by Mather [12] and by Wall [19]. Wall went on to produce counterexamples to Milnor's question. In more controlled geometric situations, positive results were obtained. In 1969, Kirby and Siebenmann [10] proved that compact TOP manifolds have the homotopy types of finite polyhedra, and in 1974 West [20], building on work of Chapman [1] and Miller [13], proved that compact ANRs have the homotopy types (actually, preferred simple homotopy types) of finite polyhedra. In 1975 Edwards and Geoghegan produced compacta shape dominated by finite CW complexes which are not shape equivalent to finite CW complexes. Siebenmann's 1965 thesis is also relevant to this problem.

We will answer Question 1 by proving the following general theorem:

THEOREM 1. *If X is compact and $d: X \rightarrow Y$ is a homotopy domination, then Y is homotopy equivalent to some compact space.*

This means, in particular, that each of Wall's examples is homotopy equivalent to some compactum and the answer to Question 1 is "no."

The work of West mentioned above allowed the notion of simple homotopy type to be extended from the category of finite CW complexes to the category of compact ANRs. It is natural to ask whether the notion extends farther, perhaps to the category of compact metric spaces. Recall that a map $f: X \rightarrow Y$ is *cell-like* if it is surjective, proper, and each point-inverse has the shape of a point. The reader unfamiliar with shape theory can safely replace "has the shape of a point" by "is contractible." The most prevalent idea for extending simple homotopy theory to compacta is to define cell-like maps to be simple. This leads to our second question.

QUESTION 2. *If Z is compact, K and L are finite polyhedra, and $r_1: Z \rightarrow K$, $r_2: Z \rightarrow L$ are cell-like, must K be simple homotopy equivalent to L ?*

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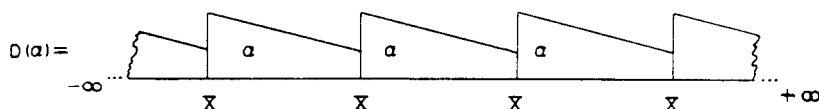
THEOREM 2. *If $f: X \rightarrow Y$ is a homotopy equivalence between compact spaces then there exist a compact space Z and cell-like maps (actually, contractible retractions, which will be defined later) $r_1: Z \rightarrow K$, $r_2: Z \rightarrow L$ such that $f \circ r_1 = r_2$.*

This shows that one cannot define a reasonable simple homotopy theory for compacta by this method. To prove Theorem 2 we imitate [4] (see also [17]) to construct the "Whitehead group" of a compact space. We then describe an infinite repetition trick which shows that these Whitehead groups are zero. This proves Theorem 2.

Two remarks are in order. First, the spaces constructed in Theorems 1 and 2 are not locally connected. Questions 1 and 2 are still open for locally connected spaces. Second, the analog of Theorem 2 for shape theory is false. See [8] for a counter-example.

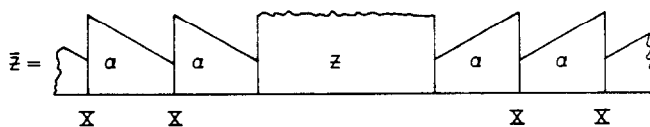
The proof of Theorem 1 is modelled on Siebenmann's variation of West's proof that compact ANRs have the homotopy types of finite complexes. We adapt appropriate techniques from simple homotopy theory and Hilbert cube manifold theory for use in more general spaces. Here is an outline of our proof:

(i) Following Mather [12], we prove that if $d: X \rightarrow Y$ is a homotopy domination with right inverse u , then Y is homotopy equivalent to a certain space $D(\alpha)$. Let $\alpha = u \circ d: X \rightarrow Y$. Then $D(\alpha)$ is the space obtained by gluing together infinitely many copies of $M(\alpha)$, the mapping cylinder of α .



(ii) We define the notion of a contractible retraction, which strengthens the notion of cell-like map. We use an infinite repetition trick to prove Theorem 2 and its generalization to proper homotopy equivalences between locally compact spaces. The proof of (i) yields a self proper homotopy equivalence of $D(\alpha)$ which reverses the ends, thus there exist a space Z and proper contractible retractions $r_1, r_2: Z \rightarrow D(\alpha)$ such that $r_2 \circ (r_1)^{-1}$ reverses the ends of $D(\alpha)$.

(iii) We use r_1, r_2 , and Z to construct a new space \bar{Z} proper homotopy equivalent to Z (and therefore homotopy equivalent to Y) which "looks like $D(\alpha)$ on the ends and like Z in the middle." Here is a schematic picture of \bar{Z} :



A compactum homotopy equivalent to \bar{Z} is easily obtained by collapsing mapping cylinders on the ends. This is the step which mimics Siebenmann's proof of West's theorem. We remark that this method of turning a problem involving dominations into a problem involving cell-like maps is a geometric analog of the well-known injection $\tilde{K}_0(Z(G)) \rightarrow Wh(Z(G \times T))$. More precisely, the end reversing self homotopy equivalence of $D(\alpha)$ covers an orientation reversing self homotopy equivalence of the mapping torus of α . This is the desired homotopy equivalence corresponding to the domination.

The rest of the paper extends these results on simple homotopy and finiteness

obstructions for compacta to higher simple homotopy in the sense of Hatcher[9]. A *PL fibration* is a PL map between finite polyhedra which is a Hurewicz fibration. An *ANR fibration* is a map from a compact absolute neighborhood retract to a finite polyhedron which is a Hurewicz fibration. Hatcher[9] has shown that different fibers of PL fibrations over connected base spaces have the same simple homotopy type. The analogous result for ANR fibrations was proved in[3] using results of Edwards, Miller, and West. It therefore makes sense to define classifying spaces $B_{PL}(K)$ and $B_{ANR}(K)$ for PL fibrations and ANR fibrations with fibers simple homotopy equivalent to K . It is known that the natural map $B_{PL}(K) \rightarrow B_{ANR}(K)$ is a homotopy equivalence. (See[9] for a near-statement of this result.) This says that the category of ANRs is equivalent to the category of polyhedra.

We attack the analogous problem for compacta. A *compact fibration* is a Hurewicz fibration with compact total space and polyhedral base. If X is compact, let $B_C(X)$ be the classifying space for compact fibrations with fiber homotopy equivalent to X and let $B_G(X)$ be the classifying space for general Hurewicz fibrations with fiber homotopy equivalent to X .

THEOREM 3. *The natural map $B_C(X) \rightarrow B_G(X)$ is a homotopy equivalence. Thus, the category of compacta is equivalent to the homotopy category.*

§1. CONTRACTIBLE RETRACTIONS

In this section we will introduce the notion of a contractible retraction (CR map) and establish our notation. By *space*, we will mean a Hausdorff space. The term *map* will denote a continuous function between spaces. A map $f: X \rightarrow Y$ is said to be *proper* if $f^{-1}(K)$ is compact for each compact $K \subset Y$. By a *homotopy* we will mean a map $f: X \times I \rightarrow Y$ and we will write $f_t: X \rightarrow Y$ for the restriction to the t th level. We will write $f_t: g = h$ if f is a homotopy with $f_0 = g$ and $f_1 = h$. If $p: X \rightarrow B$ and $q: Y \rightarrow B$ are maps, a map $f: X \rightarrow Y$ is said to be *sliced over B* if $q \circ f = p$. A *proper homotopy* is a homotopy which is a proper map. A *sliced homotopy* is a homotopy $f: X \times I \rightarrow Y$ such that each f_t is a sliced map. If X and Y are spaces, then Y^X will denote the space of continuous functions from X to Y with the compact-open topology.

Definition 1.1. Let X and Y be spaces. We say that $r: X \rightarrow Y$ is a *CR map* if there exist a map $i: Y \rightarrow X$ and a homotopy $R: X \times I \rightarrow X$ such that $r \circ i = id_Y$, $R: id = i \circ r$, $r \circ R_t = r$ for all t , and $R_t|_i(Y) = id_{i(Y)}$ for all t . We say that r is a *proper (sliced) CR map* if r and R are proper (sliced).

Note that the homotopy R contracts each point-inverse $r^{-1}(y)$ in itself to $i(y)$. The notion of CR map is a refinement of the notions of CE map[11] and shrinkable map[5]. The notion of CR map appears without a name in Theorem 5 of[18]. If $r: X \rightarrow Y$ is a CR map, we write $X \searrow Y$ or $Y \nearrow X$. When we wish to be specific, we will write $X \xrightarrow{i} Y$ or $Y \xrightarrow{j} X$. In this context, i will always denote the appropriate inclusion.

Examples.

- (1) $id: X \rightarrow X$ is a CR map.
- (2) $proj: X \times I \rightarrow X \times \{0\}$ is a CR map.
- (3) If $f: X \rightarrow Y$ is a homotopy equivalence and Z is the mapping path fibration of f , then we have $X \xrightarrow{i_1} Z \xrightarrow{i_2} Y$ such that $r_2 \circ i_1 = f$. See p. 99[19].

LEMMA 1.2. *If $r: X \rightarrow Y$ and $s: Y \rightarrow Z$ are CR maps, then $s \circ r: X \rightarrow Z$ is a CR map. If r and s are proper (sliced) CR maps, then $s \circ r$ is also a proper (sliced) CR map.*

Proof. The proof is easy and is left to the reader. ■

Here are some other easy lemmas, the proofs of which are also left to the reader. Since the definition of CR map implies that $i: Y \rightarrow i(Y)$ is a homeomorphism, we will identify Y with $i(Y)$ for the remainder of this paper.

LEMMA 1.3. *If $r: X \rightarrow Y$ is a CR map, $A \subset Y$ is closed, and $f: A \rightarrow Z$ is a map, then the induced map $X \underset{f}{\cup} Z \rightarrow Y \underset{f}{\cup} Z$ is a CR map. If r is a proper (sliced) CR map and f is proper (sliced), then the induced CR map is also proper (sliced).* ■

LEMMA 1.4. *If $X \searrow Y$ and $Y \nearrow Z$, then $X \nearrow X \underset{Y}{\cup} Z \searrow Z$, where $X \underset{Y}{\cup} Z$ is the quotient of $X \amalg Z$ obtained by identifying the two copies of Y .* ■

Similarly, if we have a sequence of CR maps

$$X_0 \nearrow X_1 \searrow \dots \nearrow X_{2n-1} \searrow X_{2n},$$

then

$$X_0 \nearrow ((X \underset{X_2}{\cup} X_3) \underset{X_4}{\cup} X_5 \dots) \underset{X_{2n-2}}{\cup} X_{2n-1} \searrow X_{2n}.$$

Thus, any sequence of CR maps and inclusions inverse to CR maps can be replaced by a single inclusion and a single CR map.

§2. A CALCULUS OF MAPPING CYLINDERS

In this section, we will adapt several more lemmas from simple homotopy theory to our more general setting.

By the *mapping cylinder* $M(f)$ of a map $f: X \rightarrow Y$, we will mean the space obtained from $X \times [0, 1] \amalg Y$ by identifying $(x, 1)$ with $f(x)$. All mapping cylinders will be given the quotient topology. If $A \subset X$ is a closed set, then the *reduced mapping cylinder* $M_A(f)$ is formed by collapsing the rays $\{a\} \times [0, 1]$, $a \in A$ to $f(A)$. If $f|_A$ is an imbedding, we will identify X with $X \times \{0\} \subset M_A(f)$; in any case, we identify Y with $Y \subset M_A(f)$.

Here is another easy lemma.

LEMMA 2.1. *If $f: X \rightarrow Y$ and $A \subset X$ is closed, then there is a CR map $r: M_A(f) \rightarrow Y$ such that $r|_X = f$. If f is proper (sliced), then r is a proper (sliced) CR map.*

Proof. r is simply the mapping cylinder collapse. ■

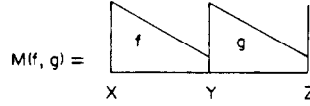
LEMMA 2.2. *If $A \subset X$ is closed and $f: X \times J \rightarrow X$ is a homotopy such that $f_t|_A = f_0|_A$ for all t , then there is a space Z_A such that $M_A(f_0) \nearrow Z_A \searrow M_A(f_1)$. If f is proper (sliced) then the CR maps produced are proper (sliced) CR maps.*

Proof. If $A = \emptyset$, let $Z = M(f)/\sim$, where $(x, s, 0) \sim (x, s', 0)$ for all $(x, s, 0) \in X \times J \times I \subset M(f)$. In other words, Z is obtained by crushing the top of the mapping cylinder to a single copy of X . The proof is standard (§6 [4]). There are CR maps from $J \times I$ to $J \times \{0, 1\} \cup \{1\} \times I$ and to $J \times \{0, 1\} \cup \{0\} \times I$ which induce the desired CR maps $Z \rightarrow M(f_0)$ and $Z \rightarrow M(f_1)$.

If $A \neq \emptyset$, form Z_A from Z by identifying all points $(a, s, t) \in A \times J \times I$ with

$f_0(a) \in Y$. The CR maps described for $A = \emptyset$ induce CR maps $M_A(f_0) \nearrow Z_A \searrow M_A(f_1)$. The verification of the sliced and/or proper version of the lemma is routine. ■

If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are maps, then $M(f, g)$ will denote the double mapping cylinder obtained from $M(f) \coprod M(g)$ by identifying $Y \subset M(f)$ with $Y \subset M(g)$.



LEMMA 2.3. If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are maps, then there is a space W such that $M(g \circ f) \nearrow W \searrow M(f, g)$. If f and g are proper (sliced) then the CR maps produced are proper (sliced).

Proof. Let $c: M(f) \rightarrow Z$ be the composition of the mapping cylinder collapse with g . The desired space W is $M(c)$. Since $c|_Y = g$, $M(f, g) = M(f) \cup M(c|_Y)$. Since $c|_X = g \circ f$, $M(g \circ f) = M(c|_X)$.

If $x \in X$, there is a triangle in $M(c)$ spanned by x , $f(x)$, and $g \circ f(x)$. The CR map $M(c) \searrow M(f, g)$ is obtained by collapsing each such triangle to the sides $[x, f(x)] \cup [f(x), g \circ f(x)]$. The CR map $M(c) \searrow M(g \circ f)$ is obtained by sliding $M(g)$ down its rays to Z and extending linearly while fixing X . The details are left to the reader. ■

COROLLARY 2.4. If $f: X \times I \rightarrow Y$ is a homotopy, then there exist maps $H: M(f_0) \rightarrow M(f_1)$, $G: M(f_1) \rightarrow M(f_0)$, and homotopies $L: id \simeq G \circ H$, $K: id \simeq H \circ G$ such that $H|(X \cup Y) = id$, $G|(X \cup Y) = id$, $L_t|(X \cup Y) = id$ for all t , and $K_t|(X \cup Y) = id$ for all t . If f is sliced (proper) then the maps and homotopies constructed above are also sliced (proper). If A is a closed subset of X and $f_t|_A = f_0|_A$ for all t , then analogous maps and homotopies between $M_A(f_0)$ and $M_A(f_1)$ can be constructed.

Proof. By Lemma 2.2, there is a space W such that $M(f_0) \nearrow W \searrow M(f_1)$. If R^0 and R^1 are the homotopies associated to the CR maps r_0 and r_1 , then the desired maps and homotopies are defined by the formulas: $H = r_1 \circ i_0$, $G = r_0 \circ i_1$, $L(x, t) = r_0 \circ R^1(i_0(x), t)$, and $K(x, t) = r_1 \circ R^0(i_1(x), t)$. Note that all maps and homotopies defined above are the identity on $X \cup Y$. ■

COROLLARY 2.5. If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are maps, then there exist maps $H: M(g \circ f) \rightarrow M(f, g)$, $G: M(f, g) \rightarrow M(g \circ f)$ and homotopies $L: id \simeq G \circ H$, $K: id \simeq H \circ G$ such that $H|(X \cup Z)$, $G|(X \cup Z)$, $K_t|(X \cup Z)$, and $L_t|(X \cup Z)$ are the identity for all t . If f and g are proper (sliced) then the maps and homotopies can be constructed to be proper (sliced).

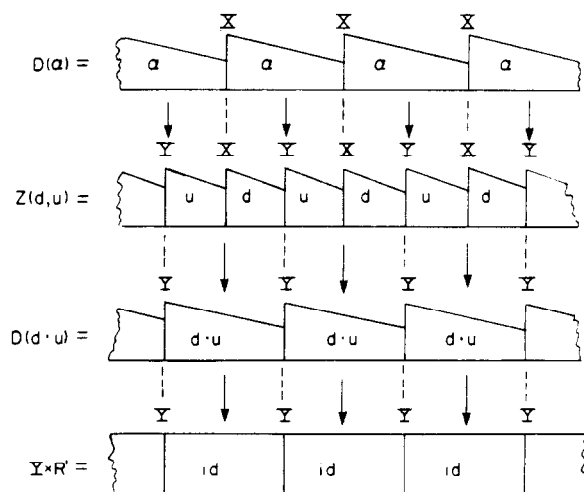
Proof. This follows from Lemma 2.3 exactly as Corollary 2.4 followed from Lemma 2.2. ■

§3. SOME BASIC FACTS ABOUT MAPPING CYLINDERS

If $d: X \rightarrow Y$ is a homotopy domination with right inverse u and $\alpha: u \circ d: X \rightarrow X$, let $D(\alpha)$ be the space formed from $\prod_{i=-\infty}^{\infty} X \times [i, i+1]$ by identifying $(x, i) \in X \times [i-1, i]$ with $(\alpha(x), i) \in X \times [i, i+1]$ for each $x \in X$ and $-\infty < i < \infty$.

PROPOSITION 3.1 [12]. If $d: X \rightarrow Y$ is a homotopy domination as above, then $D(\alpha)$ is homotopy equivalent to Y .

Proof. The proof is obtained from the following picture:



The space $Z(d, u)$ is obtained by piecing together infinitely many copies of $M(u)$ and $M(d)$ in the manner pictured. The vertical arrows represent homotopy equivalences constructed using Corollaries 2.4 and 2.5. These homotopy equivalences are the identity along the dotted lines. ■

Remark. Note that Proposition 3.1 provides a beautiful geometric affirmative solution to Whitehead's question 1. Mather's original construction is somewhat less elaborate than the one given here and may be preferred by many readers. This gives a retraction $\tau: Z(f, g) \rightarrow M(f)$. If X is a strong deformation retract of $Z(f, g)$ by h_t , then $\tau h_t|_M(f)$ is a strong deformation retraction from $M(f)$ to X .

COROLLARY 3.2. *If $f: X \rightarrow Y$ is a homotopy equivalence, then X is a strong deformation retract of $M(f)$. If f is a proper (sliced) homotopy equivalence, then the strong deformation retraction is proper (sliced).*

Proof. Let $g: Y \rightarrow X$ be a homotopy inverse for f and form a space $Z = Z(f, g)$ as in the proof of Proposition 3.1. The proof of Proposition 3.1 shows that each copy of Y in $Z(f, g)$ is a strong deformation retract of $Z(f, g)$. One performs the deformation in $Y \times R^1$ and lifts it to $Z(f, g)$. Similarly, each copy of X is a strong deformation retract of $Z(f, g)$.

A proof of this corollary can also be constructed using the homotopy extension theorem.

If X is compact and $\alpha: X \rightarrow X$ is a map, then $D(\alpha)$ is a two-ended space. Here is a corollary to the proof of Proposition 3.1 which will be needed in the sequel.

COROLLARY 3.3. *If $d: X \rightarrow Y$ is a homotopy domination with right inverse u , X is compact, and $\alpha = u \circ d$, then there is a proper homotopy equivalence from $D(\alpha)$ to $D(\alpha)$ which reverses the ends.*

Proof. Let $H: D(\alpha) \rightarrow Y \times R^1$ and $K: Y \times R^1 \rightarrow D(\alpha)$ be the homotopy equivalences constructed in proving Proposition 3.1. Let $T: Y \times R^1 \rightarrow Y \times R^1$ be the involution $T(y, t) = (y, -t)$. Then $K \circ T \circ H$ is the desired proper homotopy equivalence. ■

§4. CONSTRUCTING COMPACTA

In this section we will prove Theorem 1. The first lemma is adapted from [17]. See also §6 of [4].

LEMMA 4.1. *If X is a strong deformation retract of Y , then there exist spaces Z and W containing X such that $Z \cup_X Y \nearrow W \searrow X$. If the strong deformation retraction (both the retraction and the homotopy) is proper (sliced) then the CR maps constructed are proper (sliced).*

Proof. Let $D: Y \rightarrow X$ be the retraction and let Z be the space obtained from $M_X(D) \coprod M_X(D)$ by identifying the two copies of Y . (See [4] for a picture.) Since $M_X(D) \cup_X Y$ is the reduced mapping cylinder of $i \circ D$ and $i \circ D$ is homotopic to the identity rel X , we have $M_X(D) \cup_X Y \nearrow W' \searrow M_X(id_Y)$. Of course, W' is just the space obtained in Lemma 2.2 from the reduced mapping cylinder of the homotopy $id \simeq i \circ D$. These CR maps are the identity on the copy of Y at the top of $M_X(D)$. So we have $Z \cup_X Y \nearrow W' \cup_Y M_X(D) \searrow M_X(id_Y) \cup_Y M_X(D)$, where the unions are taken along the top copies of Y .

This last space is simply $M_X(D)$ and the CR map $M_X(D) \searrow X$ completes the proof. ■

The next proposition generalizes Theorem 2.

PROPOSITION 4.2. *If $f: X \rightarrow Y$ is a proper (sliced) homotopy equivalence, then there exist a space V and proper (sliced) CR maps $r_0: V \rightarrow X$, $r_1: V \rightarrow Y$ such that $r_1 \circ i_0 = f$.*

Proof. Let P be the space $\left\{ \frac{1}{n} \mid n \in \mathbb{Z}^+ \right\} \cup \{0\}$ with the topology inherited from \mathbb{R}^1 .

By Corollary 3.2, X is a strong deformation retract of $M(f)$. By Lemma 4.1 there are spaces Z and W and proper (sliced) CR maps such that $M(f) \cup_X Z \nearrow W \searrow X$. Let $\text{proj}: X \times P \rightarrow X$ be the projection and construct spaces $M(f)^* = (M(f) \times P) \cup_X X$, $Z^* = (Z \times P) \cup_X X$, and $W^* = (W \times P) \cup_X X$.

Crossing the CR maps of Lemma 4.1 with the identity gives proper (sliced) CR maps $(M(f) \times P) \cup_X (Z \times P) \nearrow W \times P \searrow X \times P$ which induce proper (sliced) CR maps: $M(f)^* \cup_X Z^* \nearrow W^* \searrow X$. Note, however, that $M(f)^* \cup_X M(f)$ is homeomorphic to $M(f)^*$. Thus we have:

$$X \nearrow W^* \searrow M(f)^* \cup_X = M(f)^* \cup_X Z^* \nearrow M(f) \nearrow W^* \cup_X M(f) \searrow M(f) \searrow Y.$$

Lemma 1.4 supplies the desired V . Since every map except the last collapse is the identity on X , the composition is f . ■

The space V which we have constructed is not locally connected. Proposition 4.2 plays the role in our theory that Miller's theorem [13] plays in the study of ANRs. We now prove Theorem 1.

Proof (Theorem 1). Let X be compact and let $d: X \rightarrow Y$ be a homotopy domination with right inverse u . Let $\alpha = u \circ d$ and let $f: D(\alpha) \rightarrow D(\alpha)$ be the end-reversing proper homotopy equivalence of Corollary 3.3. We will refer to the range copy of $D(\alpha)$ as $I(\alpha)$ to avoid confusion. Thus,

$$I(\alpha) = \dots \xrightarrow{\alpha} X \xrightarrow{\alpha} X \xrightarrow{\alpha} X \xrightarrow{\alpha} X \xrightarrow{\alpha} \dots \rightarrow +\infty$$

and $f: D(\alpha) \rightarrow I(\alpha)$ preserves the ends.

By the preceding proposition, there exist a space Z and proper CR maps $r_0: Z \rightarrow D(\alpha)$, $r_1: Z \rightarrow I(\alpha)$ such that $r_1 \circ i_0 = f$. Our plan is to use these CR maps to construct a space \bar{Z} proper homotopy equivalent to Z which looks like $D(\alpha)$ near $-\infty$, $I(\alpha)$ near $+\infty$, and Z in the middle. The desired compactum is a strong deformation retract of \bar{Z} via mapping cylinder collapses on the ends. See the introduction for a picture.

Choose closed neighborhoods A and B of $-\infty$ in $D(\alpha)$ and $+\infty$ in $I(\alpha)$ such that $r_0^{-1}(A) \cap r_1^{-1}(B) = \emptyset$. Form \bar{Z} by identifying points in $r_0^{-1}(A)$ with their images in A and points in $r_1^{-1}(B)$ with their images in B . Thus, \bar{Z} is the pushout:

$$\begin{array}{ccc} r_0^{-1}(A) \cup r_1^{-1}(B) & \hookrightarrow & Z \\ r_0 \cup r_1 \downarrow & & \downarrow q \\ A \cup B & \longrightarrow & \bar{Z}. \end{array}$$

We must define a proper homotopy inverse for the quotient map q . Choose a map $\rho_1: Z \rightarrow [0, 1]$ such that $\rho_1(r_0^{-1}(A)) = 0$ and $\rho_1(r_1^{-1}(B)) = 1$. Let $\rho_2: [0, 1] \rightarrow [0, 1]$ be a map such that $\rho_2(0) = 1$, $\rho_2(1) = 1$, and $\rho_2(t) = 0$ for $t \in [\frac{1}{3}, \frac{2}{3}]$. The composition $\rho = \rho_2 \circ \rho_1: Z \rightarrow [0, 1]$ is equal to 1 on $q^{-1}(A \cup B)$ and is equal to zero on a band which separates $q^{-1}(A)$ and $q^{-1}(B)$.

Let $R^0: id \simeq i_0 \circ r_0$ and $R^1: id \simeq i_1 \circ r_1$ be the homotopies guaranteed by the definition of CR maps. We define a map $v': Z \rightarrow Z$ by the formula

$$v'(z) = \begin{cases} R^0(z, \rho(z)) & \rho_1(z) \leq \frac{1}{2} \\ R^1(z, \rho(z)) & \rho_1(z) \geq \frac{1}{2}. \end{cases}$$

This is well-defined since $\rho_1(z) = (1/2)$ implies $\rho(z) = 0$, which implies that $R^0(z, \rho(z)) = z = R^1(z, \rho(z))$. It is therefore continuous. If $r_0(z) \in A$, then $\rho(z) = 1$ and $R^0(z, \rho(z)) = i_0 \circ r_0(z)$. This implies that v' is constant on the point-inverses $r_0^{-1}(a)$, $a \in A$. Similarly, v' is constant on $r_1^{-1}(b)$, $b \in B$. Thus, v' is constant on point-inverses of q and induces a map $v: \bar{Z} \rightarrow Z$. The composition $v \circ q: Z \rightarrow Z$ is clearly v' , which is homotopic to the identity via the homotopy V defined by the formula:

$$V(z, s) = \begin{cases} R^0(z, \rho(z) \cdot s) & \rho(z) \leq \frac{1}{2} \\ R^1(z, \rho(z) \cdot s) & \rho(z) \geq \frac{1}{2} \end{cases}$$

where $(z, s) \in Z \times I$.

Now consider $q \circ v: \bar{Z} \rightarrow \bar{Z}$. Since $v = v' \circ q^{-1}$, we have $q \circ v = q \circ v' \circ q^{-1}$, so the diagram below commutes.

$$\begin{array}{ccc} Z & \xrightarrow{\quad} & Z \\ q \downarrow & & \downarrow q \\ \bar{Z} & \xrightarrow{q \circ v} & \bar{Z}. \end{array}$$

If $r_0(z) \in A$, then $r_0 \circ V_s = r_0(z)$ for all $s \in [0, 1]$. Similar considerations over B show that $V_s(z)$ is constant on point-inverses of q for all s . This means that V induces a homotopy between id and $q \circ v$. This completes the proof of Theorem 1. ■

The compactum we have constructed is not locally connected. Otherwise it is not

much worse than the dominating space W . In particular, if X is n -dimensional, then the compactum constructed is $(n + 4)$ -dimensional. The argument of [7] shows that if Y is homotopy dominated by an n -dimensional CW complex $n \geq 3$ then Y has the homotopy type of an n -dimensional compactum. Of course, that argument relies on Wall's algebraic theory.

It is clear that the real key to Theorem 1 is the infinite repetition trick in Proposition 4.2. The fact that Wall's obstruction is an element of $\tilde{K}_0[Z\pi, Y]$ leads one to suspect that some infinite process is involved in every example of the sort we have constructed. It is difficult to see how such a process could be carried out using locally simply connected spaces. (We know of no locally connected examples, but we believe that their existence is less unlikely.)

CONJECTURE. *If Y is a locally simply connected compactum and Y is homotopy dominated by a finite CW complex, then Y has the homotopy type of a finite CW complex.*

A theorem confirming this conjecture would be a strong generalization of the main theorem of [20] and would presumably lead to a reasonably simple homotopy theory for locally simply connected compacta. It is not difficult to modify the construction of [7] to obtain $(n - 2)$ -connected n -dimensional compacta which are shape dominated by finite complexes and which are not shape equivalent to finite complexes.

It has been suggested that one should disprove the conjecture by shrinking down the copy of W and 0 in the proof of Proposition 4.2. To date, such constructions have failed to yield continuous maps and/or homotopies.

§5. THE PROOF OF THEOREM 3

Theorem 3 is a consequence of the following propositions.

PROPOSITION 5.1. *If $p_1: E_1 \rightarrow B$ and $p_2: E_2 \rightarrow B$ are compact fibrations and $f: E_1 \rightarrow E_2$ is a sliced homotopy equivalence, then there exist a compact fibration E_3 and sliced CR maps $E_1 \nearrow E_3 \searrow E_2$ whose composition is f .*

Proof. The sliced version of Proposition 4.2 provides a space E_3 and the desired CR maps. The only catch is that we must verify that the map $p_3: E_3 \rightarrow B$ is a Hurewicz fibration. Fortunately, the results necessary for this verification are contained in two papers of McAuley [14, 18]. In particular, [14] contains a proof that the mapping cylinder of a sliced map between Hurewicz fibrations is a Hurewicz fibration. The details are left to the reader. ■

We will need another proposition of McAuley.

PROPOSITION 5.2. *If $p_1: E_1 \rightarrow B$ and $p_2: E_2 \rightarrow B$ are compact fibrations and $r: E_1 \rightarrow E_2$ is a sliced CR map, then the natural projection $p: M(r) \rightarrow B \times I$ is a compact fibration.*

Proof. This is Theorem 5 of [18]. ■

Our next proposition answers a question posed in p. 612 [18].

PROPOSITION 5.3. *If (E_1, p_1, B) and (E_2, p_2, B) are compact fibrations and $f: E_1 \rightarrow E_2$ is a fiber homotopy equivalence, then there exists a compact fibration $(E_3, p_3, B \times I)$ such that $E_3|_{B \times \{0\}} = E_1$, $E_3|_{B \times \{1\}} = E_2$, and such that the characteristic fiber homotopy equivalence from E_1 to E_2 induced by E_3 is f .*

Proof. By Proposition 5.1, there are fibered CR maps $E_1 \xrightarrow{i_1} \bar{E} \xrightarrow{r_2} E_2$ such that $r_2 \circ i_1 = f$. By Proposition 5.2, we can glue the mapping cylinders $M(r_1)$ and $M(r_2)$ together along \bar{E} to obtain the desired E_3 . ■

We can now prove Theorem 3.

Proof. The homotopy fiber of the classifying map $B_C(X) \xrightarrow{\gamma} B_G(X)$ consists of pairs (x, w) where x is a point of $B_C(X)$ and w is a path from $\gamma(x)$ to the basepoint of $B_G(X)$. Thus, a map from S^k into the homotopy fiber consists of a Hurewicz fibration E over $S^k \times I$ which is compact over $S^k \times \{0\}$ and trivialized over $S^k \times 1$. We must extend this to a Hurewicz fibration over $D^{k+1} \times I$ which is compact over $D^{k+1} \times \{0\}$ and trivialized (extending the given trivialization) over $D^{k+1} \times \{1\}$.

Extend over $D^{k+1} \times \{1\}$ in the obvious fashion. The fibration over $S^k \times I \cup D^{k+1} \times \{1\}$ induces a trivialization $f: E|S^k \times \{0\} \rightarrow X \times S^k$. By Proposition 5.3, there is a compact concordance from $E|S^k \times \{0\}$ to $X \times S^k$ so that the induced trivialization of $E|S^k \times \{0\}$ is f . Since $X \times S^k$ extends to $X \times D^{k+1}$, we can extend our original E to a Hurewicz fibration over $D^{k+1} \times \{0, 1\} \cup S^k \times I$. By construction, this fibration is fiber homotopically trivial. According to results of P. McAuley [14, 18] and Langston, a fiber-homotopically trivial fibration over S^{k+1} extends to a fibration over D^{k+2} . This completes the proof. ■

Remark. Note that *a priori* $B_G(X)$ and $B_C(X)$ classify fibrations up to concordance. The result of Langston–Tulley quoted above and our Proposition 5.3 show that the classification up to concordance is the same as the classification up to fiber-homotopy equivalence. The interested reader can use Proposition 5.3 to give an easy direct geometric proof that a Hurewicz fibration over a CW complex with fiber homotopy equivalent to a compactum X is fiber-homotopy equivalent to a compact fibration. ■

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Institute for Advanced Study

Princeton, NJ 08540, U.S.A.

and

The University of Kentucky

Lexington, KY 40506, U.S.A.